Stiffness method - Matrix/computer analysis

The Slope-Deflection method generated a series of simultaneous equations, which had to be solved.

This method assumed that the axial extensions of members was very small

We can extend this to include axial stiffnesses/extensions and can write the equations as a matrix equation:

Stiffness method - Matrix/computer analysis

{External Loads} = [Stiffness coefficients]
x {Joint Displacements}

where { } is a n x 1 column vector and [ ] is a n x n matrix

We can then invert the stiffness matrix to find the unknown displacements.

Matrix inversion is very tedious to do manually, but lends itself to computer solution.

Stiffness method - Matrix/computer analysis

Procedure :

1. Formulate the stiffness matrix for each member, relating forces and displacements at member ends. This is known as the local stiffness matrix.

2. Combine the local matrices using conditions of equilibrium and compatibility to give a global stiffness matrix for the whole structure.

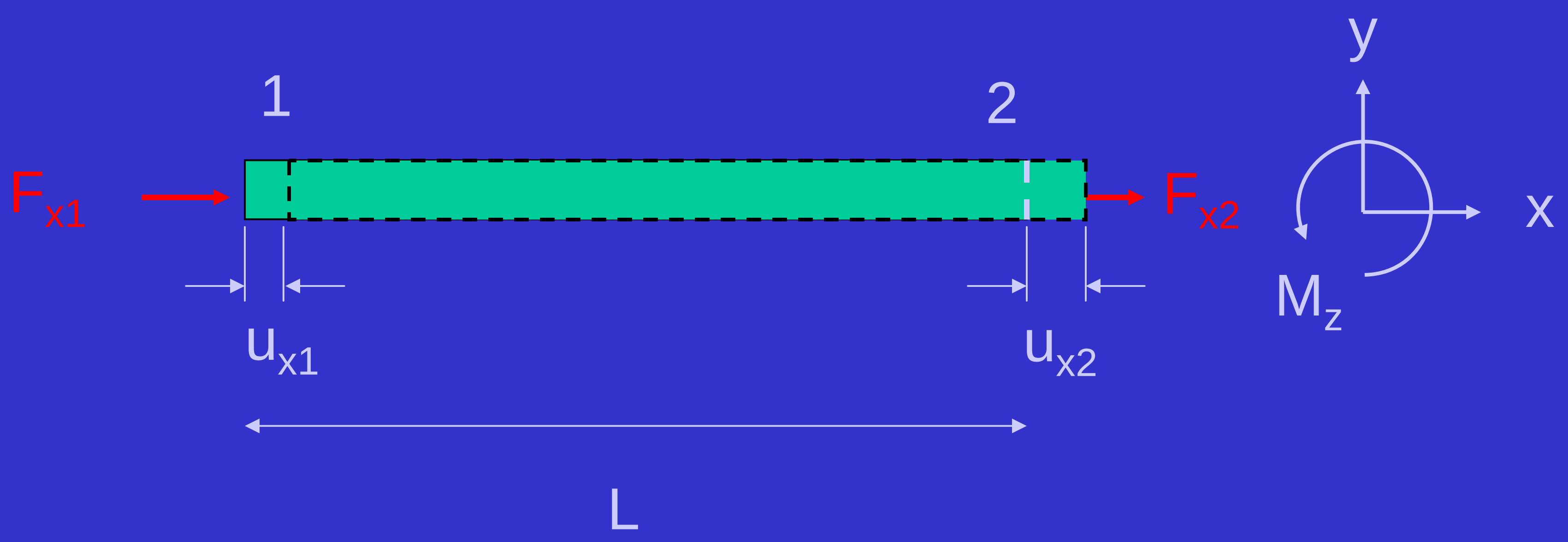
3. Invert the stiffness matrix to find the unknown displacements.

4. Substitute displacements back into local stiffness matrix to find member forces.

Stiffness method - Matrix/computer analysis

We can demonstrate this method using a simple example of an axially loaded member.

1. Formulate the local stiffness matrix.



Now consider each displacement separately:

$$F_{x1} = \frac{AE}{L} u_{x1} \qquad F_{x2} = -\frac{AE}{L} u_{x1} \qquad F_{x1} = -\frac{AE}{L} u_{x2} \qquad F_{x2} = \frac{AE}{L} u_{x2}$$

Stiffness method - Matrix/computer analysis

Use Principle of Superposition to find forces when displacements act simultaneously.

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$$

where k = AE/L

Leading diagonal terms relate forces and displacements at same end and in same direction.

Stiffness method - Matrix/computer analysis

Use Principle of Superposition to find forces when displacements act simultaneously.

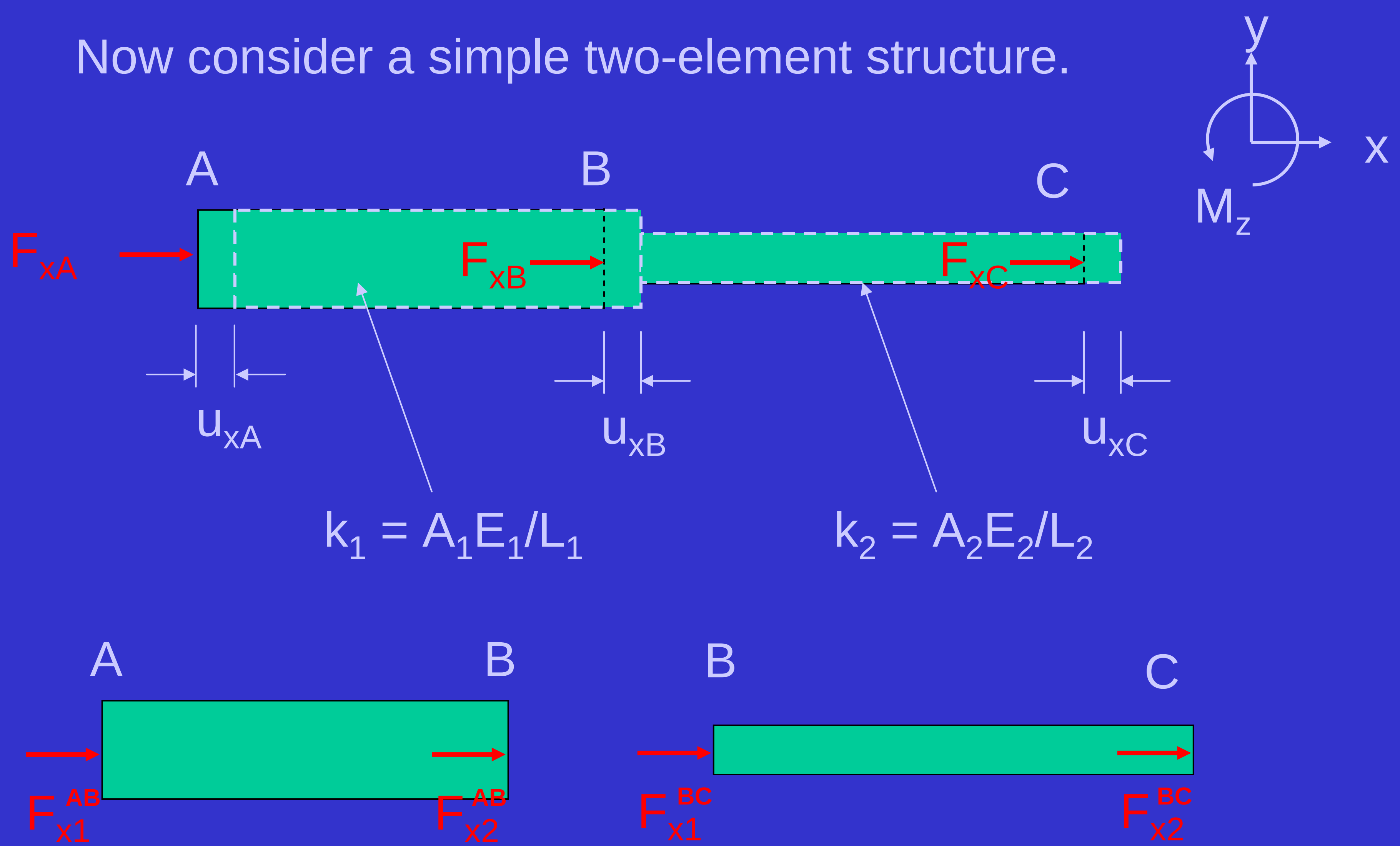
$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$$

where k = AE/L

Off-diagonal terms relate forces and displacements to cross-coupled ends

Stiffness method - Matrix/computer analysis

Now consider a simple two-element structure.



Stiffness method - Matrix/computer analysis

For AB: local stiffness matrix

$$\begin{Bmatrix} F_{x1}^{AB} \\ F_{x2}^{AB} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_{x1}^{AB} \\ u_{x2}^{AB} \end{Bmatrix}$$

For BC: local stiffness matrix

$$\begin{Bmatrix} F_{x1}^{BC} \\ F_{x2}^{BC} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{x1}^{BC} \\ u_{x2}^{BC} \end{Bmatrix}$$

1. Formulate the local stiffness matrices.

Stiffness method - Matrix/computer analysis

Compatibility :

$$u_{x1}^{AB} = u_{xA}$$

$$u_{x2}^{AB} = u_{x1}^{BC} = u_{xB}$$

$$u_{x2}^{BC} = u_{xC}$$

$$\begin{Bmatrix} F_{x1}^{AB} \\ F_{x2}^{AB} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x1}^{BC} \\ F_{x2}^{BC} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{xB} \\ u_{xC} \end{Bmatrix}$$

2. Use equilibrium and compatibility to find global stiffness matrix

Stiffness method - Matrix/computer analysis

Equilibrium :

$$\begin{Bmatrix} F_{xA} \\ F_{xB} \\ F_{xC} \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{x2} \\ 0 \end{Bmatrix}_{AB} + \begin{Bmatrix} 0 \\ F_{x1} \\ F_{x2} \end{Bmatrix}_{BC}$$

Hence

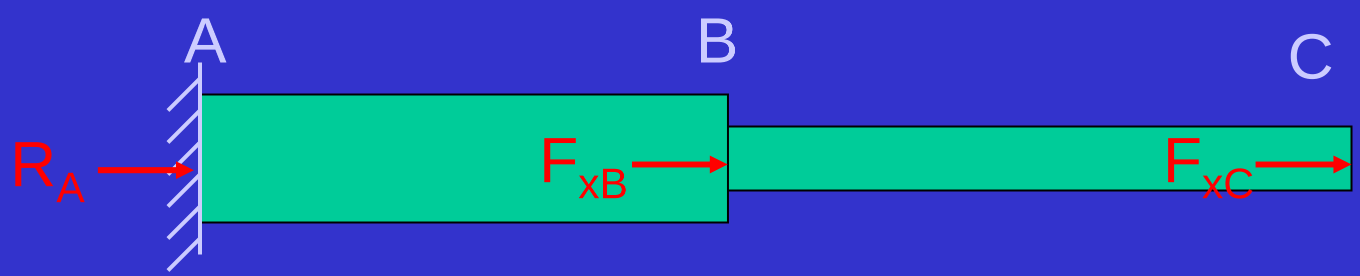
$$\begin{Bmatrix} F_{xA} \\ F_{xB} \\ F_{xC} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{xA} \\ u_{xB} \\ u_{xC} \end{Bmatrix}$$

2. Use equilibrium and compatibility to find global stiffness matrix

Stiffness method - Matrix/computer analysis

We cannot yet invert the stiffness matrix because it is singular. i.e. we will get 0 = 0 !

This is because the structure has no Boundary Conditions. It is floating in space…!



If we fix the structure at A :

$$F_{xA} = R_A \qquad \text{and} \quad u_{xA} = 0$$

3. Invert the stiffness matrix to find the unknown displacements.

Stiffness method - Matrix/computer analysis

$$\begin{Bmatrix} R_A \\ F_{xB} \\ F_{xC} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{xB} \\ u_{xC} \end{Bmatrix}$$

The 1st column of the stiffness matrix is multiplied by zero & plays no part. The stiffness matrix reduces to:

$$\begin{Bmatrix} F_{xB} \\ F_{xC} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{xB} \\ u_{xC} \end{Bmatrix}$$

3. Invert the stiffness matrix to find the unknown displacements.

Stiffness method - Matrix/computer analysis

Inverting the stiffness matrix gives :

$$\begin{Bmatrix} u_{xB} \\ u_{xC} \end{Bmatrix} = 1/k_1k_2 \begin{bmatrix} k_2 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} F_{xB} \\ F_{xC} \end{Bmatrix}$$

Hence u_{xB} & u_{xC} are found

3. Invert the stiffness matrix to find the unknown displacements.